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Supersymmetry and Superfields in Three Euclidean Dimensions

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The simplest supersymmetry algebra and superspace in three-dimensional Euclidean (3dE) space is examined. Representations of the algebra are considered and the implications of restricting the space of states to states with positive definite norm are determined. A superspace is defined and superfields are introduced. Supersymmetric field theory models in 3dE are described in both superfield and component field forms. The relationship between these models and similar models in four-dimensional Minkowski space is described.

KEY WORDS: supersymmetry; superfields; Euclidean dimension.

1. INTRODUCTION

Supersymmetry (SUSY) has been widely discussed in four-dimensional Minkowski space (4dM) (Bailin and Love, 1994; Gates *et al.*, 1983; Wess and Bagger, 1992; West, 1990); it is anticipated that SUSY will eventually be shown to be a fundamental symmetry of nature. However, the structure of the SUSY algebra depends on the nature of the space in which it is defined. A discussion of the SUSY algebra in four-dimensional Euclidean Space (4dE or $4 + 0$ space) can be found in Lukierski and Nowicki (1983, 1984) and McKeon and Sherry (2000, 2001); the SUSY algebra in $2 + 2$ dimensions is analyzed in Brandt *et al.* (2000) and Ketov *et al.* (1992, 1993a,b). Scalar models in three-dimensional Minkowski space (3dM or $2 + 1$ space) are introduced in Gates *et al.* (1983) and Siegel (1979), while some of their quantum properties are elucidated in Dilkes *et al.* (1997), Gates *et al.* (1983), McKeon and Nguyen (1999), and McKeon and Portelance-Gagné (in press). Other SUSY algebras and models are analyzed in Kugo and Townsend (1983) and McKeon (2000a).

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There are several motivations for examining Euclidean field theories. Vacuum tunneling by means of instantons involves working in Euclidean space. The high temperature limit of a $(D+1)$-dimensional Minkowski space field theory is effectively a D-dimensional Euclidean space field theory. This motivates a study of supersymmetric theories in Euclidean spaces. This is in contrast to the use of Euclidean space as a calculational tool to evaluate Feynman integrals (obtained by a Wick rotation). One can generalize the Wick rotation to relate SUSY theories in Euclidean spaces to SUSY theories in Minkowski spaces (McKeon and Sherry, 2000, 2001; Mehta, 1992; Nicolai, 1978; Van Nieuwenhuizen and Waldron, 1996).

In this paper we propose to analyze the SUSY algebra and formulate supersymmetric models directly in three-dimensional Euclidean space (3dE or $3+0$ dimensions) much as we have done in 4dE (McKeon and Sherry, 2000, 2001). The nature of SUSY in $3+0$ dimensions is quite different from that of SUSY in $2+1$ dimensions. In $2+1$ dimensions Majorana spinors can occur (Gates *et al.*, 1983; Kugo and Townsend 1983; McKeon, 2000a) while it is well known that in $3+0$ dimensions they cannot. Thus in $2+1$ dimensions one can choose to use Majorana spinorial generators in the SUSY algebra (Gates *et al.*, 1983), while in $3+0$ dimensions, as is well known, the spinorial generators must be Dirac spinors (Kugo and Townsend, 1983; McKeon, 2000a). As a result the simplest self-conjugate SUSY algebra in 3dE is more akin to an $N=2$ SUSY algebra in 3dM than an $N=1$ version; furthermore, a central charge necessarily occurs.

When analyzing the representations of this algebra it is found that if the states are to have a positive definite norm then the magnitude of the momentum must be less than the central charge: the central charge yields on *upper* bound on the momentum (Salam and Strathdee, 1974a). An important consequence of this result is that unlike the situation in 4dM or, indeed, 3dM, it is not possible to set the central charge to zero. The central charge *must* be included in the treatment of the SUSY algebra in 3dE. This is independent of the model being considered.

We also construct a superspace for the SUSY algebra in $3+0$ dimensions. We recall that in the 4dM superspace (Bailin and Love, 1994; Gates *et al.*, 1983; Wess and Bagger, 1992; West, 1990) a 2-component Weyl spinor and its conjugate (equivalently a 4-component Majorana spinor) act as Grassmann coordinates, while in the 3dM superspace (Gates *et al.*, 1983) only one 2-component Majorana spinor Grassmann coordinate is required. This simpler superspace structure simplifies the construction of supersymmetric models in 3dM: for example, a scalar superfield automatically yields an irreducible representation of the SUSY algebra. In contrast, the scalar superfield in 4dM yields a reducible representation of the SUSY algebra; irreducible representations can be obtained by imposing conditions, such as chirality, antichirality, or reality, on the superfield. The situation in 3dE is more akin to that in 4dM than that in 3dM. The Grassmann coordinates in the 3dE superspace must be Dirac spinors so that there are two independent

2-component spinorial components θ and $\theta^\dagger$. This causes the component field expansion of the scalar superfield to be very similar to the equivalent expansion in 4dM.

Following the approach of Sohnius (1978) to the central charge in the $N = 2$ 4dM SUSY algebra, the realization of the generators of the SUSY algebra on this superspace involves the introduction of a bosonic coordinate conjugate to the central charge.

The analysis of scalar superfields follows quite closely the well-known analysis in 4dM (Bailin and Love, 1994; Gates *et al.*, 1983; Wess and Bager, 1992; West, 1990). Chiral and antichiral scalar superfields can be defined. The corresponding component fields are two complex spin-0 fields and a complex spin-$\frac{1}{2}$ field. Supersymmetric interacting models like the Wess–Zumino model in 4dM can be written down in both superfield and component field form. Real scalar superfields are also introduced. The component field expansion involves a 3-vector field. We introduce a superfield generalization of $U(1)$ gauge transformations and construct a supersymmetric $U(1)$ gauge theory analogous to super QED in 4dM.

The results that we find for the chiral and real scalar superfields, and the field theory models constructed using them, are very reminiscent of the corresponding results in 4dM. The similarities are most obvious when the 4dM examples are written in 2-component form. We find that the relationship is not a simple-minded dimensional reduction in which coordinates are merely discarded as in Brink *et al.* (1977) and McKeon and Sherry (2000, 2001); such a procedure does not lead to central charges and so cannot reproduce the correct SUSY algebra in 3dE. Rather, we show that the SUSY algebra in 3dE is the particular subalgebra of the SUSY algebra in 4dM in which "boost" generators are discarded and momentum in the time-like direction becomes the central charge. The outcome of this reduction procedure is quite similar to Siegel's mass dependent central charge introduced in Siegel (1980) in the context of 3dM. We, however, cannot discard the dependence on the bosonic coordinate conjugate to the central charge as the central charge forms an upper bound on the momentum.

The remainder of this paper is organized as follows. In Section 2 we examine the SUSY algebra in 3dE; we consider the representations of this algebra and the role played by the central charges. In Section 3 we construct the 3dE superspace and develop our treatment of chiral superfields. In Section 4 we consider the real superfields and the supersymmetric $U(1)$ gauge theory. Section 5 contains a discussion of the relationship between the SUSY algebras in 4dM and 3dE and the consequential relationships between the models constructed in each case. In Section 6 we consider the $N = 1$ and $N = 2$ SUSY algebras in 3dM. We examine the role of the central charge in the $N = 2$ case and note the similarity with the SUSY algebra in 3dE. In Appendix A we include the conventions and notations which we have used in our calculations in 3dE. Appendix B deals with a supersymmetry algebra in 3dE in which the commutator $[P^a, P^b]$ is taken to be nonzero.

2. THE SUSY ALGEBRA IN 3dE

The simplest representation of the Dirac gamma matrices in 3dE is provided by the standard Pauli spin matrices $\tau^a = \tau^{a\dagger}$, $a = 1, 2, 3$. Charge conjugation is defined in terms of a matrix C for which

$$C_{\tau^a} C^{-1} = -\tau^{a\mathrm{T}}; \tag{1}$$

the explicit form of C (up to a phase factor which we set equal to 1) is

$$C = \tau^2 = C^\dagger = -C^\mathrm{T} = -C^* = C^{-1}. \tag{2}$$

In $3 + 0$ dimensions a 2-component SO(3) spinor ψ has Hermitian conjugate

$$\psi^\dagger \, (= \bar{\psi}) \tag{3}$$

and charge conjugate

$$\psi_\mathrm{C} = C_{\bar{\psi}}^{T} = \tau^2 \psi^*. \tag{4}$$

ψ and its charge conjugate ψ_C transform in exactly the same manner under SO(3): if

$$\psi \to e^{i\vec{\omega}\cdot\vec{\tau}/2}\psi = U\psi \tag{5}$$

then, by (1) and (4),

$$\psi_\mathrm{C} \to U\psi_\mathrm{C}. \tag{6}$$

From (2) and (4) it is evident that $(\psi_\mathrm{C})_\mathrm{C} = -\psi$, and hence, as is well known, it is not possible to impose a Majorana condition in 3dE equating ψ and ψ_C. Spinors in 3dE must be Dirac; at best a pair of Dirac spinors ψ_1 and ψ_2 may together satisfy symplectic Majorana conditions $(\psi_1)_\mathrm{C} = \psi_2$ and $(\psi_2)_\mathrm{C} = \psi_1$ (Kugo and Townsend, 1983; McKeon, 2000a).

In this paper we only treat self-conjugate SUSY algebras: if a generator O belongs to the algebra, then so also does its conjugate $O^\dagger$. In this context, the simplest possible supersymmetric extension of the Poincaré algebra in 3dE (ISO(3)) involves Dirac spinorial generators R and $R^\dagger$ and a central charge Z:

$$\{R, R^\dagger\} = \vec{\tau}\cdot\vec{P} + Z = \begin{bmatrix} Z + P^3, & P^1 - iP^2 \\ P^1 + iP^2, & Z - P^3 \end{bmatrix}, \tag{7a}$$

$$\{R, R\} = \{R^\dagger, R^\dagger\} = 0, \tag{7b}$$

$\vec{P}$ being the translation generator in 3dE.

Following the standard approach used in $3 + 1$ dimensions (Salam and Strathdee, 1974a; Sohnius, 1978) we can rewrite the SUSY anticommutator in terms of Fermionic creation and annihilation operators. In this way information about the representations of the SUSY algebra can be deduced. Working in the reference frame in which $\vec{P} = (0, 0, P)$ we see that the anticommutator (7) is equivalent to the anticommutators

$$\{R_1, R_1^\dagger\} = P + Z, \tag{8a}$$

$$\{R_2, R_2^\dagger\} = -P + Z, \tag{8b}$$

$$\{R_1, R_2^\dagger\} = \{R_1^\dagger, R_2\} = 0, \tag{8c}$$

where $R = [{R_1 \atop R_2}]$.

Equations (8a) and (8b) are crucial in determining the nature of the states in an irreducible representation of the SUSY algebra. To begin with, we note that negative norm states will necessarily occur if $|p| > Z > 0$. Prohibiting the occurrence of negative norm states means

$$|P| \leq Z, \tag{9}$$

i.e., the central charge acts as an *upper* bound on the momentum of the state. This feature has been noted previously in $4 + 0$ dimensions (McKeon and Sherry, 2000, 2001). It is quite different to what pertains in $3 + 1$ dimensions, where the central charge, if it occurs, acts as a lower bound on the magnitude of the 4-momentum.

Following from (9) we see that in 3dE the central charge is *not* allowed to vanish; it must be present to guarantee nontriviality. A vanishing central charge will ensure that only zero momentum states can exist. This feature also was present in $4 + 0$ dimensions (McKeon and Sherry, 2000, 2001) but is very different to what pertains in $3 + 1$ dimensions (Bailin and Love, 1994; Gates *et al.*, 1983; Wess and Bagger, 1992; West, 1990), where the central charge can be set to zero without trivializing the theory.

Saturation of the upper bound inherent in (9) will lead to a situation analogous to what happens in the $(3 + 1)$-dimensional case; there, if the BPS lower bound is saturated, then half the states in the model do not occur (Bogamolnyi, 1976; Olive and Witten, 1978; Osborne, 1979; Prasad and Sommerfield, 1975; Sohnius, 1978). In this case, only states associated with $R_1^\dagger$ (or $R_2^\dagger$) will occur depending on whether $Z = P$ (or $-P$).

In order to find the representations of the SUSY algebra (7) we proceed in the standard manner. The other commutators in the 3dE SUSY algebra are

$$[J^a, J^b] = i\epsilon^{abc} J^c, \tag{10a}$$

$$[J^a, R_i] = -\frac{1}{2}(\tau^a R)_i, \tag{10b}$$

$$[J^a, P^b] = i\epsilon^{abc} P^c, \tag{10c}$$

$$[Z, J^a] = [Z, P^a] = [Z, R_i] = [P^a, R_i] = 0. \tag{10d}$$

The operators $\vec{P}$, $\vec{J}^2$, $\vec{P}\cdot\vec{J}/\vec{P}|$, Z, R, and $R^\dagger$ are used to classify the states. As the first four of these operators commute with each other we can choose to use their simultaneous eigenstates. Essentially, R and $R^\dagger$ will act as ladder operators linking the states with different eigenvalues for $\vec{J}^2$ and $\vec{P}\cdot\vec{J}/|\vec{P}|$.

The first state we consider in our representation $|I\rangle$ will have the following properties:

$$Z|I\rangle = z|I\rangle >, \tag{11a}$$

$$\vec{P}^2|I\rangle = M^2|I\rangle >, \tag{11b}$$

$$\frac{\vec{P}\cdot\vec{J}}{|\vec{P}|}|I\rangle = m|I\rangle >, \tag{11c}$$

$$\vec{J}^2|I\rangle = j(j+1)|I\rangle >, \tag{11d}$$

$$R_i|I\rangle = 0, \quad i = 1, 2, \tag{11e}$$

where j can take on one of the values $0, \frac{1}{2}, 1, \ldots,$ and m can take on any value in the range $-j, -j+1, \ldots, j-1, j$. The other states in the representation are given by

$$|i\rangle = R_i^\dagger|I\rangle, \quad i = 1, 2, \tag{12a}$$

$$|F\rangle = R_1^\dagger R_2^\dagger|I\rangle. \tag{12b}$$

If we align $\vec{P}$ along the third axis, $\vec{P} = (0, 0, M)$, then it is easily seen that

$$J_3|I\rangle = m|I\rangle, \tag{13a}$$

$$J_3|1\rangle = \left(m + \frac{1}{2}\right)|1\rangle, \tag{13b}$$

$$J_3|2\rangle = \left(m - \frac{1}{2}\right)|2\rangle, \tag{13c}$$

$$J_3|F\rangle = m|F\rangle, \tag{13d}$$

and

$$J^2|F\rangle = j(j+1)|F\rangle, \tag{14}$$

while the states $|1\rangle$ and $|2\rangle$ are, in general, superpositions of eigenstates of the operator $\vec{J}^2$ corresponding to the eigenvalues $j \pm \frac{1}{2}$. The eigenvalue of the central charge Z is common to all the states, as Z commutes with all the other operators in the algebra. On the other hand, we have already seen that we must; have $M \leq Z$.

It is worth noting that if $|I\rangle$ corresponds to a spin-0 state, with $j = 0$, then there are only two spin-0 states and two spin-$\frac{1}{2}$ states in the representation. However, if $|I\rangle$ corresponds to a spin-$\frac{1}{2}$ state, there will be one spin-0 state, four spin-$\frac{1}{2}$ states coresponding to two doublets, and a spin-1 triplet state. In all cases the number of Fermionic states will match exactly the number of Bosonic states.

3. 3dE SUPERSPACE AND CHIRAL SUPERFIELDS

The simplest 3dE SUSY algebra, given in Eqs. (7) and (10), contains two independent Fermionic generators. As a consequence we use two independent 2-component Dirac spinor Grassmann coordinates in the 3dE superspace, namely θ and $\theta^\dagger$. In addition, during to the essential and critical occurrence of the central charge Z, we use a fourth bosonic coordinate, ζ, conjugate to Z much as in Siegel (1979) and Sohnius (1978). In this way superspace techniques will be similar to those employed in 4dM (Salam and Strathdee, 1974b, 1975). The notation and conventions used in our 3dE superspace are listed in Appendix A.

The eight coordinates in the 3dE superspace are

$$Z = \{x^a, a = 1, 2, 3; \zeta; \theta_i, i = 1, 2; \theta_i^\dagger, i = 1, 2\}. \tag{15}$$

In this space the 3dE SUSY algebra of (7) and (10) can be realized by

$$R_i = \partial_i^\dagger - \frac{i}{2}(\tau^a\theta)_i\nabla^a - \frac{i}{2}\theta_i\partial_\zeta, \tag{16a}$$

$$R_i^\dagger = \partial_i - \frac{i}{2}(\theta^\dagger\tau^a)_i\nabla^a - \frac{i}{2}\theta_i^\dagger\partial_\zeta, \tag{16b}$$

$$P^\mu = -i\nabla^a, \tag{16c}$$

$$Z = -i\partial_\zeta, \tag{16d}$$

and

$$J^a = -i\epsilon^{abc}x^b\nabla^c - \frac{1}{2}\theta^\dagger\tau^a\partial^\dagger + \frac{1}{2}\partial\tau^a\theta. \tag{16e}$$

This representation in similar to the one employed for the $N = 2$ SUSY algebra in 3dM by Sohnius (1978). The SUSY transformations generated by R_i and $R_i^\dagger$ in the 3dE superspace are

$$\delta x^a = [\xi^\dagger R - R^\dagger\xi, x^a] = -\frac{i}{2}\xi^\dagger\tau^a\theta + \frac{i}{2}\theta^\dagger\tau^a\xi, \tag{17a}$$

$$\delta\theta_k = [\xi^\dagger R - R^\dagger\xi, \theta_k] = \xi_k, \tag{17b}$$

$$\delta\theta_k^\dagger = [\xi^\dagger R - R^\dagger\xi, \theta_k^\dagger] = \xi_k^\dagger, \tag{17c}$$

$$\delta\zeta = [\xi^\dagger R - R^\dagger\xi, \zeta] = -\frac{i}{2}\xi^\dagger\theta + \frac{i}{2}\theta^\dagger\xi. \tag{17d}$$

The occurrence of derivatives with respect to ζ in R_i and $R_2^\dagger$ is crucial to realize the algebraic relations (7); it is not possible to realize the algebra unless such terms are included. The occurrence of such terms corresponding to central charges has been previously noted (Sohnius, 1978; West, 1990) in the case of $N = 2$ SUSY in 4dM.

It is possible to identify operators which anticommute with R and $R^\dagger$ and play the role of covariant derivatives for the 3dE SUSY. They are

$$D_i = \partial_i^\dagger + \frac{i}{2}(\tau^a\theta)_i\nabla^a + \frac{i}{2}\theta_i\partial_\zeta, \tag{18a}$$

$$D_i^\dagger = \partial_i + \frac{i}{2}(\theta^\dagger\tau^a)_i\nabla^a + \frac{i}{2}\theta_i^\dagger\partial_\zeta. \tag{18b}$$

We now introduce scalar superfields in our 3dE superspace. These are functions of x^a, ζ, θ, and $\theta^\dagger$ satisfying

$$\Phi'(x', \zeta', \theta', \theta^{\dagger\prime}) = \Phi(x, \zeta, \theta, \theta^\dagger). \tag{19}$$

An expansion of Φ in powers of θ and $\theta^\dagger$ can be used to identify multiplets of fields which transform among themselves under SUSY transformations. Such a representation, however, turns out to be reducible. We can impose a ‘chiral condition’ which is respected by the SUSY transformations, namely

$$D_i\Phi = 0 \tag{20}$$

or an antichiral condition

$$D_i^\dagger\Phi = 0, \tag{21}$$

as $\{D_i, Q_j\} = \{D_i, Q_j^\dagger\} = 0$. Such chiral, or antichiral, superfields involve component fields which transform in an irreducible representation of the SUSY algebra, as we demonstrate below. The expansion of a chiral scalar superfield in terms of component fields can be written in the form

$$\Phi(x^a, \zeta, \theta, \theta^\dagger) = \phi(y^a, w) + \lambda^\dagger(y^a, w)\theta + F(y^a, w)\theta_C^\dagger\theta, \tag{22}$$

where we have used

$$D_i\theta_j = 0, \tag{23a}$$

$$D_i\left(x^a - \frac{i}{2}\theta^\dagger\tau^a\theta\right) \equiv D_i y^a = 0, \tag{23b}$$

$$D_i\left(\zeta - \frac{i}{2}\theta^\dagger\theta\right) \equiv D_i w = 0. \tag{23c}$$

The component fields ϕ and F are complex scalar fields while λ is a Dirac spinor field.

The SUSY transformations of the component fields can be simply deduced; the unitary implementation of these transformations is

$$U = \exp[\xi^\dagger R - R^\dagger \xi] \tag{24}$$

so that

$$\delta\Phi = [\xi^\dagger R - R^\dagger \xi, \Phi]. \tag{25}$$

It follows that

$$\delta\phi(x,\zeta) = \lambda^\dagger(x,\zeta)\xi, \tag{26a}$$

$$\delta\lambda^\dagger(x,\zeta) = -i\xi^\dagger(\phi_{,a}(x,\zeta)\tau^a + \phi_{,\zeta}(x,\zeta)) - 2F(x,\zeta)\xi_{\mathrm{C}}^\dagger, \tag{26b}$$

$$\delta F(x,\zeta) = \frac{i}{2}(\lambda_{,a}^\dagger(x,\zeta)\tau^a\xi_{\mathrm{C}} - \lambda_{,\zeta}^\dagger\xi_{\mathrm{C}}) \tag{26c}$$

are the component field transformations.

A supersymmetric action can be written down in terms of superfields, and also in terms of component fields. A kinetic term is given by

$$S_{\mathrm{K}} = \int d^3x\, d\zeta\, d^2\theta\, d^2\theta^\dagger\, \Phi^*\Phi \tag{27}$$

and a super potential by

$$S_{\mathrm{P}} = \int d^3x\, d\zeta\, d^2\theta\, d^2\theta^\dagger\, \delta(\theta^\dagger)[m\Phi^2 + g_3\Phi^3 + g_4\Phi^4] + \mathrm{H.C.} \tag{28}$$

The component field forms of the kinetic and potential terms in the action are obtained by using the following expansion of the chiral scalar superfield Φ:

$$\begin{aligned}\Phi(y,w) = {} & \phi(x,\zeta) - \phi_{,a}(x,\zeta)\left(\frac{i}{2}\theta^\dagger\tau^a\theta\right) - \phi_{,\zeta}(x,\zeta)\left(\frac{i}{2}\theta^\dagger\theta\right)\\ & + \frac{1}{2!}\phi_{,ab}(x,\zeta)\left(\frac{i}{2}\theta^\dagger\tau^a\theta\right)\left(\frac{i}{2}\theta^\dagger\tau^b\theta\right) + \frac{1}{2!}\phi_{,\zeta\zeta}(x,\zeta)\left(\frac{i}{2}\theta^\dagger\theta\right)^2\\ & + \phi_{,\zeta a}(x,\zeta)\left(\frac{i}{2}\theta^\dagger\theta\right)\left(\frac{i}{2}\theta^\dagger\tau^a\theta\right) + \lambda^\dagger(x,\zeta)\theta\\ & - \lambda_{,a}^\dagger(x,\zeta)\theta\left(\frac{i}{2}\theta^\dagger\tau^a\theta\right) - \lambda_{,\zeta}^\dagger(x,\zeta)\theta\left(\frac{i}{2}\theta^\dagger\theta\right) + F(x,\zeta)\theta_{\mathrm{C}}^\dagger\theta\end{aligned} \tag{29}$$

and the formulae in Appendix A for integration over the Grassmann variables. We find

$$\begin{aligned}S_{\mathrm{K}} = \int d^3x\, d\zeta\, \Bigg[& -\frac{1}{16}(\nabla^2\phi^*\phi + \phi^*\nabla^2\phi - 2\phi_{,a}^*\phi_{,a}) - \frac{1}{16}(\phi_{,\zeta\zeta}^*\phi + \phi^*\phi_{,\zeta\zeta}\\ & -2\phi_{,\zeta}^*\phi_{,\zeta}) - \frac{i}{8}(\lambda_{,a}^\dagger\tau^a\lambda - \lambda^\dagger\tau^a\lambda_{,a} + \lambda^\dagger\lambda_{,\zeta} - \lambda_{,\zeta}^\dagger\lambda) + F^2\Bigg]\end{aligned} \tag{30}$$

and

$$S_p = \int d^3x\, d\zeta \left[m\left(2\phi F + \frac{1}{2}\lambda^\dagger \lambda_C\right) + g_3\left(3\phi^2 F + \frac{3}{2}\phi\lambda^\dagger\lambda_C\right) + g_4(4\phi^3 F + 3\phi^2\lambda^\dagger\lambda_C)\right] + \text{H.C.} \tag{31}$$

This model, consisting of a supersymmetric action for a pair of complex spin-0 fields and a Dirac spin-$\frac{1}{2}$ field, is the analogue of the Wess–Zumino model in 4dM. It is interesting to note, however, that the spinor mass term is of the Majorana type, as are the Yukawa couplings between the spin-0 and spin-$\frac{1}{2}$ fields.

4. THE REAL SCALAR SUPERFIELD AND A SUPERSYMMETRIC *U*(1) GAUGE MODEL

In this section we turn our attention to supersymmetric vector field theories in 3dE. In particular, we formulate a noninteracting supersymmetric $U(1)$ gauge invariant model. We follow very closely the standard methods used in 4dM.

We saw in Section 3 that imposing a chiral constraint on the scalar superfield results in a highest spin component field with spin $\frac{1}{2}$. To include spin-1 vector fields we must relax this condition. Nevertheless we can impose a reality condition on the scalar superfield, now denoted by V:

$$V = V^*. \tag{32}$$

Making use of Eqs (A8) and (A9) we find the most general form of this superfield

$$\begin{aligned} V(x,\zeta,\theta,\theta^\dagger) = {} & C + [\chi^\dagger\theta + \theta^\dagger\chi] + \theta^\dagger\tau^a\theta V^a \\ & + \theta^\dagger\theta E + (M + iN)\theta_C^\dagger\theta + (M - iN)\theta^\dagger\theta_C \\ & + (\tilde{\Lambda}^\dagger\theta + \theta^\dagger\tilde{\Lambda})\theta^\dagger\theta + (\theta^\dagger\theta)^2\tilde{D}, \end{aligned} \tag{33}$$

where the fields $C(x,\zeta)$, $E(x,\zeta)$, $M(x,\zeta)$, $N(x,\zeta)$, and $\tilde{D}(x,\zeta)$ are real spin-0 fields, $\chi(x,\zeta)$ and $\tilde{\Lambda}(x,\zeta)$ are 2-component Dirac spin-$\frac{1}{2}$ fields, and $V^a(x,\zeta)$ is a real spin-1 3-vector field. The superfield $V(x,\zeta,\theta,\theta^\dagger)$ can also be termed a vector superfield, but this identification could be misleading. However, the set of component fields correspond to a vector supermultiplet.

Just as in 4dM there is an interesting interplay between SUSY and $U(1)$ gauge transformations. For this reason we now consider the SUSY generalization of an $U(1)$ gauge transformation. A real superfield undergoes the transformation.

$$V \to V + \delta V = V + i(\Phi^* - \Phi), \tag{34}$$

where Φ is a chiral superfield and Φ^* is an antichiral superfield. In terms of component fields this corresponds to

$$\delta C = i(\phi^* - \phi), \tag{35a}$$

$$\delta_\chi = i\lambda, \tag{35b}$$

$$\delta V^a = -\frac{1}{2}(\phi + \phi^*)_{,a}, \tag{35c}$$

$$\delta E = -\frac{1}{2}(\phi + \phi^*)_{,\zeta}, \tag{35d}$$

$$\delta(M + iN) = -iF, \tag{35e}$$

$$\delta\tilde{\Lambda} = \frac{1}{2}(\vec{\tau}\cdot\vec{\nabla} - \partial_\zeta)\lambda, \tag{35f}$$

$$\delta\tilde{D} = \frac{i}{8}\left(\nabla^2 - \partial_\zeta^2\right)(\phi^* - \phi). \tag{35g}$$

A redefinition of the $(\theta^\dagger\theta)^2$, $(\theta^\dagger\theta)\theta$, and $(\theta^\dagger\theta)\theta^\dagger$ components of the real superfield gives rise to a simpler and more useful set of component field gauge transformations:

$$\tilde{\Lambda} = \Lambda - \frac{i}{2}(\vec{\tau}\cdot\vec{\nabla} - \partial_\zeta)\chi, \tag{36a}$$

$$\tilde{D} = D + \frac{1}{8}\left(\nabla^2 - \partial_\zeta^2\right)C. \tag{36b}$$

The gauge transformations for the ∇ and D component fields are then

$$\delta\Lambda = 0, \qquad \delta D = 0. \tag{37}$$

It is also evident from the form of the gauge transformations in (35) that the C, χ, M, and N fields can be transformed to zero by an appropriate choice of the chiral superfield components ϕ, λ, and F. This is the analogue of the Wess–Zumino gauge in 4dM; in this gauge the real superfield has the form

$$V_{\mathrm{WZ}} = (\theta^\dagger\tau^a\theta)V^a + (\theta^\dagger\theta)E + (\Lambda^\dagger\theta + \theta^\dagger\Lambda)\theta^\dagger\theta + (\theta^\dagger\theta)^2 D. \tag{38}$$

A field strength superfield invariant under the gauge transformation (34) is given by the chiral spinor superfield

$$W_i = (D_{\mathrm{C}}^\dagger D)D_{\mathrm{C}_i}V. \tag{39}$$

[Since $D^3 = 0$ it follows that $D_j W_i = 0$]. The component field expansion of this chiral spinor superfield is, by analogy with (22),

$$W_i = X_i(y, w) + Y_{ij}(y, w)\theta_j + Z_i(y, w)\theta_{\mathrm{C}}^\dagger\theta. \tag{40}$$

We can use (18) in (39) to relate the field strength component fields X, Y, and Z to the component fields of the real superfield in the Wess–Zumino gauge:

$$X_i(x,\zeta) = W_i\ |_{\theta=\theta t=0} = 2\Lambda_i, \tag{41a}$$

$$Y_{ij}(x,\zeta) = D_j^\dagger W_i\ |_{\theta=\theta\dagger=0} = 4D\delta_{ij} - 2\vec{\tau}\cdot(\vec{\nabla}\times\vec{V}) - 2i\vec{\tau}\cdot\vec{V}_{,\zeta} + 2i\vec{\tau}\cdot\vec{\nabla}E, \tag{41b}$$

$$Z_i(x,\zeta) = \frac{1}{4}(D^\dagger D_{\mathrm{C}})W_i\ |_{\theta=\theta\dagger=0} = i\vec{\tau}\cdot\vec{\nabla}\Lambda_{\mathrm{C}} - i\Lambda_{\mathrm{C},\zeta}. \tag{41c}$$

A suitable supersymmetric and $U(1)$ gauge invariant kinetic term in the action for the real superfield V is given by

$$S_{\mathrm{K}} = \int d^3x\, d\zeta\, d^2\theta\, d^2\theta^\dagger\, \delta(\theta^\dagger)(W_i^\dagger)_{\mathrm{C}} W_i + \mathrm{H.C.} \tag{42}$$

Making use of (40) and (41) above we can write this kinetic term in terms of the component fields in the Wess–Zumino gauge

$$S_{\mathrm{K}} = \int d^3x\, d\zeta\, [\Lambda_{\mathrm{C}}^\dagger\vec{\tau}\cdot\vec{\nabla}\Lambda_{\mathrm{C}} - \Lambda_{\mathrm{C}}^\dagger\Lambda_{\mathrm{C},\zeta} + 4D^2 \\ - (\vec{\nabla}\times\vec{V} + i\vec{V}_{,\zeta} - i\vec{\nabla}E)^2] + \mathrm{H.C.} \tag{43}$$

This form of the kinetic term in manifestly gauge invariant under the $U(1)$ gauge transformations of (35c), (35d), and (37); it is also invariant under SUSY transformations of the component fields in the Wess–Zumino gauge followed by a gauge transformation (as the SUSY transformations do not respect the Wess–Zumino gauge condition).

We next find the SUSY transformations of the component fields of the real scalar superfield V by examining

$$\delta V = [\xi^\dagger R - R^\dagger\xi, V]. \tag{44}$$

The resulting transformations for the "unshifted" component fields are

$$\delta C = \xi^\dagger\chi + \chi^\dagger\xi, \tag{45a}$$

$$\delta_\chi = \vec{\tau}\cdot\vec{V}\xi + E\xi + 2(M - iN)\xi_{\mathrm{C}} + \frac{i}{2}\vec{\tau}\cdot\vec{\nabla}C\xi + \frac{i}{2}C_{,\zeta}\xi, \tag{45b}$$

$$\delta V^a = -\frac{1}{2}(\xi^\dagger\tau^a\tilde{\Lambda} + \tilde{\Lambda}^\dagger\tau^a\xi) - \frac{i}{4}(\chi_{,b}^\dagger\tau^a\tau^b\xi - \xi^\dagger\tau^b\tau^a\nabla_\chi^b \\ + \chi_{,\zeta}^\dagger\tau^a\xi - \xi^\dagger\tau^a\chi_{,\zeta}), \tag{45c}$$

$$\delta E = \frac{1}{2}(\xi^\dagger\tilde{\Lambda} + \tilde{\Lambda}^\dagger\xi) - \frac{i}{4}(\chi_{,a}^\dagger\tau^a\xi - \xi^\dagger\vec{\tau}\cdot\vec{\nabla}_\chi + \chi_{,\zeta}^\dagger\xi - \xi^\dagger\chi_{,\zeta}), \tag{45d}$$

$$\delta(M + iN) = \frac{1}{2}\xi^\dagger\tilde{\Lambda}_{\mathrm{C}} - \frac{i}{4}(\xi^\dagger\vec{\tau}\cdot\vec{\nabla}\chi_{\mathrm{C}} + \xi^\dagger\chi_{\mathrm{C},\zeta}), \tag{45e}$$

$$\delta\tilde{\lambda} = 2\tilde{D}\xi + \frac{i}{2}(\vec{\tau}\cdot\vec{\nabla} + \partial_\zeta)E\xi - \frac{i}{2}\tau^a(\vec{\tau}\cdot\vec{\nabla}\partial_\zeta)V^a\xi$$
$$+ i(\partial_\zeta - \vec{\tau}\cdot\vec{\nabla})(M - iN)\xi_C, \tag{45f}$$

$$\delta\tilde{D} = \frac{i}{4}(\xi^\dagger\vec{\tau}\cdot\vec{\nabla}\tilde{\Lambda} + \xi^\dagger\tilde{\Lambda}_{,\zeta}) + \text{H.C.} \tag{45g}$$

The transformations in terms of the shifted fields will result in alterations to (45c), (45d), (45e), (45f), and (45g), as follows:

$$\delta V^a = \frac{i}{2}\xi^\dagger\nabla^a\chi - \frac{1}{2}\xi^\dagger\tau^a\Lambda + \text{H.C.}, \tag{46a}$$

$$\delta E = \frac{i}{2}\xi^\dagger\Lambda + \frac{i}{2}\xi^\dagger\chi_{,\zeta} + \text{H.C.}, \tag{46b}$$

$$\delta(M + iN) = \frac{1}{2}\xi^\dagger\Lambda_C - \frac{i}{2}\xi^\dagger(\vec{\tau}\cdot\vec{\nabla} + \partial_\zeta)\chi_C, \tag{46c}$$

$$\delta\Lambda = 2D\xi + i\vec{\tau}\cdot(\vec{\nabla}E - \vec{V}_{,\zeta} + i\vec{\nabla}\times\vec{V})\xi, \tag{46d}$$

$$\delta D = \frac{i}{4}\xi^\dagger(\vec{\tau}\cdot\vec{\nabla} + \partial_\zeta)\Lambda + \text{H.C.} \tag{46e}$$

From these SUSY transformations, it is evident that one can also define an irreducible "curl" multiplet (such as in 4dM) by considering Λ, $\Lambda^\dagger$, D, $V_{a,b} - V_{b,a}$, and $V_{b,\zeta} - E_{,b}$.

It is clear from either form of the component field SUSY transformations that the Wess–Zumino gauge form of the superfield V_{WZ} is not respected. However, if one considers a SUSY transformation of V_{WZ} followed by a $U(1)$ gauge transformation, one can choose the gauge transformation so as to restore the Wess–Zumino gauge form of the superfield. The chiral superfield components of the gauge transformation are chosen to satisfy

$$\phi - \phi^* = 0, \tag{47a}$$

$$i\lambda + (\vec{\tau}\cdot\vec{V}_{WZ} + E_{WZ})\xi = 0, \tag{47b}$$

$$-iF + \frac{1}{2}\Lambda^\dagger_{WZ}\xi_C = 0. \tag{47c}$$

With this choice of gauge transformation the effective SUSY transformations in the Wess–Zumino gauge are

$$\delta_{\text{eff}}E = \frac{1}{2}(\xi^\dagger\Lambda + \Lambda^\dagger\xi) - \phi_{,\zeta}, \tag{48a}$$

$$\delta_{\text{eff}}V^a = -\frac{1}{2}\xi^\dagger\tau^a\Lambda - \frac{1}{2}\Lambda^\dagger\tau^a\xi - \nabla^a\phi, \tag{48b}$$

$$\delta_{\text{eff}}\Lambda = 2D\zeta + i\vec{\tau}\cdot(\vec{\nabla}E - \vec{V}_{,\zeta} + i\vec{\nabla}\times\vec{V})\xi, \tag{48c}$$

$$\delta_{\text{eff}}D = \frac{i}{4}\xi^{\dagger}(\vec{\tau}\cdot\vec{\nabla} + \partial_{\zeta})\Lambda + \text{H.C.}, \tag{48d}$$

where E, V^a, Λ, and D are understood to be Wess–Zumino gauge fields while ϕ is a real scalar field.

Coupling the real vector superfield to chiral "matter" superfields can be done as in 4dM; in addition, one can also generalize the discussion to accommodate nonabelian gauge symmetries.

5. RELATIONSHIP BETWEEN SUSY$_{3+1}$ AND SUSY$_{3+0}$

It is clear from the results of the previous sections that the 3dE supersymmetric models constructed bear more than a passing resemblance to the equivalent models in 4dM. We now examine this relationship more closely, beginning with the SUSY algebras. The relationship between SUSY in $3+1$ dimensions and $2+1$ dimensions has been considered in Siegel (1979, 1980).

The usual SUSY$_{3+1}$ algebra consists of the ISO(3,1) algebra

$$[P_\mu, P_\nu] = 0, \tag{49a}$$

$$[M_{\mu\nu}, P_\lambda] = i(\eta_{\nu\lambda}P_\mu - \eta_{\mu\lambda}P_\nu), \tag{49b}$$

$$[M_{\mu\nu}, M_{\lambda\sigma}] = i(\eta_{\nu\lambda}M_{\mu\sigma} + \eta_{\mu\sigma}M_{\nu\lambda} - \eta_{\mu\lambda}M_{\nu\sigma} - \eta_{\nu\sigma} - M_{\mu\lambda}), \tag{49c}$$

together with the defining commutators for the 2-spinor charge S_α

$$[M_{\mu\nu}, S_\alpha] = -i(\sigma_{\mu\nu})_\alpha{}^\beta S_\beta, \tag{50a}$$

$$[P_\mu, S_\alpha] = 0, \tag{50b}$$

similar commutation relations for $\bar{S}_{\dot\alpha}$ and the anticommutators

$$\{S_\alpha, S_\beta\} \neq \{\bar{S}_{\dot\alpha,}\bar{S}_{\dot\beta}\} = 0, \tag{51a}$$

$$\{S_\alpha, \bar{S}_{\dot\beta}\} = 2\sigma^\mu{}_{\alpha\dot\beta}P_\mu. \tag{51b}$$

[We use the notation and conventions of Bailin and Love (1994)]

If we now exclude only the $M_{0\alpha}$, $a = 1, 2, 3$, generators from the ISO(3,1) algebra we are left with an ISO(3) $\times$ $U(1)$ subalgebra

$$[P_a, P_b] = [P_a, P_0] = 0, \tag{52a}$$

$$[M_{ab}, P_{\rm c} = i(\delta_{ac}P_b - \delta_{ab}P_{\rm c}), \tag{52b}$$

$$[M_{ab}, P_0] = 0, \tag{52c}$$

$$[M_{ab}, M_{cd}] = -i(\delta_{ad}M_{bc} + \delta_{bc}M_{ad} - \delta_{ac}M_{bd} - \delta_{bd}M_{ac}), \tag{52d}$$

where P_0 is the generator of $U(1)$ and P_a, M_{ab} are the generators of ISO(3). The 2-spinor SUSY charges S_α and $\bar{S}_{\dot\alpha}$ can be rewritten in terms of the SU(2) subalgebra of ISO(3). Nothing that

$$\sigma_{ab} = -\frac{i}{2}\epsilon_{abc}\tau^{\mathrm{c}} \tag{53}$$

and defining

$$J^a = \frac{1}{2}\epsilon^{abc}M_{bc}, \tag{54}$$

Eq. (50a) for $\mu = a$, $\nu = b$ gives

$$[J^a, S_\alpha] = -\frac{1}{2}(\tau^a S)_\alpha. \tag{55}$$

As the SU(2) subalgebra of ISO(3) resides equally in the two SU(2) subalgebras of ISO(3,1), when we restrict our attention of ISO(3) it is no longer necessary to use dotted and undotted indices. The various types of spinors related to S_α can be identified as follows:

$$S_\alpha{}^* = \bar{S}_{\dot\alpha}, \tag{56a}$$

$$S^\alpha = \epsilon^{\alpha\beta}S_\beta, \tag{56b}$$

$$\bar{S}^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\bar{S}_{\dot\beta}, \tag{56c}$$

with $\epsilon^{\alpha\beta} = \epsilon^{\dot\alpha\dot\beta} = -iC$. We can now map the SO(3,1) 2-spinors to SU(2) spinors as follows:

$$S_\alpha \rightarrow \sqrt{2}R_i, \tag{57a}$$

$$\bar{S}_{\dot\alpha} \rightarrow \sqrt{2}R_i^\dagger, \tag{57b}$$

$$\bar{S}^{\dot\alpha} \rightarrow -i\sqrt{2}R_{\mathrm{C}i}, \tag{57c}$$

$$S^\alpha \rightarrow i\sqrt{2}R_{\mathrm{C}i}^\dagger, \tag{57d}$$

where the subscript C on R_{C} indicates the charge conjugate defined as in (4), and where we have replaced the α-index by an i-index. Nothing further that

$$\sigma^\mu = (1, \vec{\tau}) \tag{58}$$

and identifying

$$P^\mu = (Z, \vec{P}), \tag{59}$$

we see that the anticommutators (51a) and (51b) of the SUSY_{3+1} algebra can be rewritten as

$$\{R, R\} = \{R^\dagger, R^\dagger\} = 0, \tag{60a}$$

$$\{R, R^\dagger\} = \vec{r}\cdot\vec{P} + Z. \tag{60b}$$

The subalgebra of the $SUSY_{3+1}$ algebra whose bosonic part is ISO(3) $\times$ $U(1)$ consists of Eqs. (52), (55), and (60) together with

$$[\vec{P}, R] = [Z, R] = 0. \tag{61}$$

This algebra is identical to the 3dE SUSY algebra we examined in Section 2.

The mappings of Eq. (57) can be used more generally to establish the connection between 4dM spinors and 3dE spinors. This will provide a connection between the model constructed in Section 3 and the Wess–Zumino model in 4dM, and between the $U(1)$ gauge invariant model of Section 4 and super QED in 4dM. In the latter case one must also identify the scalar field E with the temporal component A_0 of the 4-vector field in 4dM. In this identification, Z is identified with P^0, so that it is clear that the bosonic coordinate ζ, associated with the central charge Z, is identified with t; $\frac{\partial}{\partial\zeta}$ is identified with $\frac{\partial}{\partial t}$. However, it must be noted that the boost operators M^{0a} have been removed from the algebra.

The relationship between the $SUSY_{3+1}$ models and the $SUSY_{3+0}$ models is similar to, but quite distinct from, the way in which dimensional reduction was used in Brink *et al.* (1977) where an $N = 2$ SUSY model in 4dM is obtained from a SUSY model in 6dM. [In Mckeon and Sherry (2000, 2001), the Zumino model (Zumino, 1977) in 4dE is obtained by this form of dimensional reduction.] The central charge Z and its associated bosonic coordinate ζ disturb the link between the two approaches. Let us recall how dimensional reduction works in Brink *et al.* (1977). Essentially the fields in the higher dimensions are set independent of the extra dimensions and all generators in the algebra which relate to the extra dimensions in any way are discarded. In each case the extra dimensions are simply eliminated; there are no apparent central charges in the models. Another feature of this approach is that each of the resulting models ($N = 2$ SUSY in 4dM and the Zumino model) are *on-shell* SUSY models without auxiliary fields. In contrast, in our case we relate off-shell supersymmetric models in 4dM to *off-shell* supersymmetric models in 3dE, and the necessary central charges are included.

If instead of the approach used here we were to use the form of dimensional reduction employed in Brink *et al.* (1977) to go from $3 + 1$ dimensions to $3 + 0$, it would be tantamount to dropping the ζ-dependence in all of the fields. Essentially this would mean $[Z, \text{field}] = 0$ for each field in the model. If all the fields have zero eigenvalue for Z, then the states associated with these fields must also have zero eigenvalue for Z. But, as we have seen in Section 2, the Z eigenvalue for a state acts as an *upper* bound on the momentum of that state. Such a model would have no actual content: all states would be zero momentum states and would have zero norm. In fact, there would be no states.

Central charges can occur in $N > 1$ SUSY algebras in 4dM. Superspace approaches to such extended SUSY models have been developed. In standard superspace approaches it proves very difficult to construct off-shell supersymmetric models. In such approaches arguments can be given to set the dependence of all fields on the bosonic coordinates associated with the central charges to zero (West, 1990). Such arguments hinge on the fact that $\{D, D\}$ yields Z while $\{D, \bar{D}\}$ yields P in 4dM. In 3dE, as seen from (18), $\{D, D^\dagger\}$ is zero while $\{D, D'^\dagger\}$ yields both P and Z. This difference ensures that the 4dM arguments do not apply to the cases considered in this paper. We must retain the ζ-dependence if we want the full SUSY algebra, including central charges, implemented in nontrivially on superfields in the models.

It is worth nothing in this context that the approach to $N = 2$ SUSY in 4dM followed in Ohta *et al.* (1986), where harmonic superspace is used, involves extra dimensions associated with the central charges, and that these extra dimensions are compactified, not ignored.

As we have retained the ζ-dependence in our fields, and thereby provided representations of the full SUSY algebra in 3dE, the actions in our two models involve integrations over ζ. Thus a "fourth" dimension is introduced. This feature will clearly have implications for the quantization and renormalizability of such models.

6. SUSY ALGEBRAS IN 3dM

We now consider SUSY algebras in 3dM. These algebras differ from the SUSY algebra in 3dE mainly because of the very different properties of spinors in 3dM. We begin by reviewing briefly these properties and contrasting with the situation in 3dE.

It is well known that Majorana spinors can be defined in 3dM (Kugo and Townsend, 1983; McKeon, 2000a). Let us investigate why this is so. We will choose as Dirac γ-matrices

$$\gamma^0 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \tau^2, \quad \gamma^1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = i\tau^1, \quad \gamma^3 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = i\tau^3 \tag{62}$$

satisfying

$$\{\gamma^a, \gamma^b\} = -2\eta^{ab}1, \quad \eta^{ab} = \mathrm{diag}(-, +, +). \tag{63}$$

Defining the Dirac conjugate and the charge conjugate of a Dirac 2-spinor ψ as

$$\bar{\psi} = \psi^\dagger A \tag{64a}$$

and

$$\psi_{\mathrm{C}} = C\bar{\psi}^T \tag{64b}$$

respectively, the matrices A and C must satisfy (Kugo and Townsend, 1983; McKeon 2000a)

$$A\gamma^\mu A^{-1} = \gamma^{\mu\dagger} \tag{65a}$$

and

$$C\gamma^{\mu}C^{-1} = -\gamma^{\mu T}. \tag{65b}$$

It is easy to see that in the above γ-matrix representation

$$C = A = \gamma^0 \tag{66}$$

can be used to solve these equations. The charge conjugate spinor is now

$$\psi_{\mathrm{C}} = CA^T\psi^{\dagger T} = -\psi^*. \tag{67}$$

As

$$(\psi_{\mathrm{C}})_{\mathrm{C}} = \psi \tag{68}$$

it follows that the Majorana condition

$$\psi = \psi_{\mathrm{C}} \tag{69}$$

can be satisfied by a 2-spinor, provided

$$\psi^* = -\psi. \tag{70}$$

The consistency of a Majorana 2-spinor in 3dM means that a self-conjugate SUSY algebra can be written down with just one spinor charge. This is the $N = 1$ SUSY algebra in 3dM. It will consist of the usual Poincaré ISO(2,1) algebra in 3dM together with a Majorana spinor generator Q satisfying the anti-commutation relation

$$\{Q, \bar{Q}\} = \gamma \cdot P. \tag{71}$$

Just as in 4dM, it is not possible to include a central charge in this anticommutator.

However, a central charge can be included if we consider an $N = 2$ SUSY algebra with two Majorana spinor generators $Q_\alpha(\alpha = 1, 2)$. They will satisfy the anticommutation relation

$$\{Q_\alpha, \bar{Q}_\beta\} = \gamma \cdot P\delta_{\alpha\beta} + iZ\epsilon_{\alpha\beta}, \tag{72}$$

where $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$, $\epsilon_{12} = +1$, and Z is the central charge operator.

We now consider the $N = 1$ and $N = 2$ SUSY algebras in terms of Fermionic creation and annihilation operators, as we did in Section 2 for the 3dE case. In the $N = 1$ case the Majorana condition (70) means that Q takes the form

$$Q = i\begin{bmatrix} q \\ p \end{bmatrix}, \tag{73}$$

where q, p are real. If we form the complex operator

$$\Lambda = q + ip \tag{74}$$

then, in the frame of reference where $\vec{P} = (M, 0, 0)$, the SUSY anticommutator (71) becomes

$$\{\Lambda, \Lambda\} = \{\Lambda^\dagger, \Lambda^\dagger = 0, \tag{75a}$$

$$\{\Lambda, \Lambda^\dagger\} = 2M, \tag{75b}$$

showing that $\Lambda^\dagger$ and Λ can be interpreted as Fermionic creation and annihilation operators respectively.

In the $N = 2$ case, the Majorana spinor generators take the form

$$Q_\alpha = i \begin{bmatrix} q_\alpha \\ p_\alpha \end{bmatrix}, \quad \alpha = 1, 2, \tag{76}$$

where q_α, p_α, $\alpha = 1, 2$, are real. Taking now the linear combinations

$$\Lambda = \frac{1}{2}[(q_1 - p_2) + i(q_2 + P_1)] \tag{77a}$$

and

$$\Omega = \frac{1}{2}[(q_1 + p_2) + i(q_2 - p_1)], \tag{77b}$$

we find the diagonalized form of the anticommutators

$$\{\Lambda, \Lambda\} = \{\Lambda^\dagger, \Lambda^\dagger = \{\Omega, \Omega\} = \{\Omega^\dagger, \Omega^\dagger\} = 0, \tag{78a}$$

$$\{\Lambda, \Lambda^\dagger\} = M - Z, \tag{78b}$$

$$\{\Omega, \Omega^\dagger\} = M + Z, \tag{78c}$$

again in the reference frame in which $\vec{P} = (M, 0, 0)$. To interpret $\Lambda^\dagger, \Omega^\dagger$ as Fermionic creation operators and Λ, Ω as Fermionic annihilation operators we must require the Hilbert space of states to be positive definite. Thus we must have either (Salam and Strathdee, 1974a; Sohnius, 1978)

$$M > |Z| \tag{79}$$

or

$$M = |Z|. \tag{80}$$

In the latter case the states associated with either $\Lambda^\dagger$ or $\Omega^\dagger$ are discarded as they are zero norm states (Bogamolnyi, 1976; Olive and Witten, 1978; Osborne, 1979; Prasad and Sommerfield, 1975). The central charge Z acts as a *lower* bound on M, as is typical of Minkowski spaces. We note that it is quite consistent to set Z to zero at the outset, unlike the situation in 3dE.

A superspace corresponding to the $N = 1$ SUSY algebra in 3dM has been discussed (Dilkes *et al.*, 1997; Gates *et al.* 1983; Mckeon and Nguyen, 1999; Mckeon and Portelance-Gagné, in press; Siegel, 1979) and is well understood.

In the notation of this paper it will consist of a Bosonic space with coordinates x^a, $a = 1, 2, 3$, and a Fermionic space with coordinates provided by the Majorana 2-spinor θ. The Majorana spinor generator Q can be realized on this superspace by

$$Q_i = \frac{\partial}{\partial \theta_i} - \frac{i}{2}(\bar{\theta}\gamma_a)_i \nabla^a \tag{81}$$

while the SUSY 'covariant derivative' D_i is realized by

$$D_i = \frac{\partial}{\partial \theta_i} + \frac{i}{2}(\bar{\theta}\gamma_a)_i \nabla^a. \tag{82}$$

A scalar superfield $\Phi(x, \theta)$ when expanded in powers of θ yields the component field expansion

$$\Phi(x, \theta) = A(x) + \bar{\lambda}(x)\theta + F(x)\bar{\theta}\theta. \tag{83}$$

It is well understood how to construct supersymmetric models in this case (Dilkes *et al.*, 1997; Gates *et al.* 1983; Mckeon and Nguyen, 1999; Mckeon and Portelance-Gagné, in press; Siegel, 1979).

In the $N = 2$ case one could propose a pair of Majorana 2-spinor coordinates θ_α, $\alpha = 1, 2$, corresponding to the Majorana 2-spinor generators Q_α, $\alpha = 1, 2$. An alternative approach would be to use a Dirac 2-spinor coordinate θ and its Dirac conjugate $\bar{\theta}$ where

$$\theta = \frac{1}{\sqrt{2}}(\theta_1 + i\theta_2) \tag{84a}$$

and

$$\bar{\theta} = \theta^\dagger \gamma^0 = \frac{1}{\sqrt{2}}(\bar{\theta}_1 + i\bar{\theta}_2). \tag{84b}$$

This would correspond to using, in place of Q_2, in the $N = 2$ SUSY algebra the Dirac 2-spinor generator R and its Dirac conjugate $\bar{R}$ defined by

$$R = \frac{1}{\sqrt{2}}(Q_1 + iQ_2), \tag{85a}$$

$$\bar{R} = R^\dagger \gamma^0 = \frac{1}{\sqrt{2}}(\bar{Q}_1 - i\bar{Q}_2). \tag{85b}$$

The $N = 2$ SUSY algebra anticommutation relation (72) takes the following form in terms of R and $\bar{R}$

$$\{R, \bar{R}\} = \gamma \cdot P + Z, \tag{86a}$$

$$\{R, R\} = \{\bar{R}, \bar{R}\} = 0. \tag{86b}$$

The similarity in structure between this form and the anticommutator (7) of the SUSY algebra in 3dE, not withstanding the use of $\bar{R}$ in (86) and $R^\dagger$ in (7), shows that a superspace can now be constructed for the $N = 2$ SUSY algebra in 3dM just as was done in Sections 2 and 3 for the SUSY algebra in 3dE. Furthermore, these $N = 2$ models in 3dM can be obtained from $N = 1$ models in 4dM by following the approach of section 5 above. One simply employs the subgroup ISO(2,1) $\times$ $U(1)$ of ISO(3,1) in place of ISO(3) $\times$ $U(1)$. This is consistent with Siegel's treatment of $N = 2$ SUSY models in 3dM (Siegel, 1979, 1980).

We note that $N = 2$ SUSY models in 3dM do not require harmonic superspace, unlike $N = 2$ SUSY models in 4dM (Galperin *et al.*, 1984) or SUSY models in 4dE (Mckeon, 2000b).

7. DISCUSSION

The main focus of this paper has been on examination of the supersymmetric extension of ISO(3), the symmetric group associated with 3dE. We have considered in Section 2 the simplest SUSY algebra in 3dE, and we saw that it must contain a central charge generator. This result is crucial; we saw that every state of the theory is subject to an upper bound on the momentum of the state, namely the eigenvalue of Z for the state. These conclusions followed very simply from the structure of the SUSY algebra in 3dE. We further investigated briefly the representations of the algebra and classified the various states in a given representation.

The SUSY algebra in 3dE is seen to involve a Dirac 2-spinor SUSY charge; thus it is more akin to an $N = 2$ SUSY algebra than an $N = 1$ SUSY algebra. Nevertheless, we developed in Section 3 a superspace approach with scalar superfields for this SUSY algebra. We wrote down a realization of each of the generators of the algebra in terms of differential operators on the superspace and we also introduced SUSY covariant derivative D and $D^\dagger$ operators. In doing this it was necessary to include in the superspace a bosonic coordinate conjugate to the central charge operator Z, denoted by ζ. Unlike the situation in 4dM, it is neither possible nor useful to project out of the theory all dependence on ζ. The fact that Z provides an upper bound on the momentum of states means that we must retain ζ-dependence.

In our superspace we introduced a scalar superfield. We found, as in 4dM, that imposing a chiral constraint $D\Phi = 0$ led to an irreducible representation of the SUSY algebra. We also constructed a supersymmetric action for the chiral superfield, including kinetic, mass, and self-interaction terms, in both superfield and component field forms. In Section 4 we saw that a real superfield could undergo $U(1)$ gauge transformations with a chiral superfield gauge function. A supersymmetric $U(1)$ gauge invariant model, analogous to super QED in 4dM, was written down, both in superfield form and in component field form in a Wess–Zumino-like gauge. In each of the cases we considered, we deduced the form of the SUSY field transformations for the component fields.

In Section 5 we discussed the relationship between the models we constructed in 3dE and the well-known Wess–Zumino and super QED models in 4dM. The relationship followed from the relationship between the corresponding SUSY algebras. In essence, the SUSY algebra in 3dE corresponds to the subalgebra of the SUSY algebra in 4dM where the Lorentz boost generators M^{oa} alone have been excised.

Two aspects, in particular, of our work will require further examination. They both concern the central charge operator Z. On the one hand, Z provides a bound for $|\vec{P}|$. The significance of this bound in a quantized theory is not clear at the moment. On the other hand, Z gives rise to an extra bosonic dimension with coordinate ζ. Again, the implications of this extra dimension is unclear for both quantization and renormalizability.

We are currently investigating a problem somewhat related to the theme of this paper. It concerns the supersymmetric extension of the Galilean group in $3+1$ dimensions. The Galilean group can be obtained by a Wigner–Inonu contraction of ISO(3,1) (Gilmore, 1974); a similar contraction of the SUSY algebra in $3+1$ may yield this extension.

APPENDIX A

In this appendix we gather together conventions concerning Dirac spinors and matrices in three-dimensional Euclidean space.

We choose to use hermitian Dirac γ-matrices, which we identify with the Pauli spin matrices,

$$\tau^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \tau^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \tau^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{A1}$$

which satisfy

$$\tau^a \tau^b = \delta^{ab} + i\epsilon^{abc}\tau^c, \quad a, b, c, = 1, 2, 3. \tag{A2}$$

We identify the charge conjugation matrix as

$$C = \tau^2, \tag{A3}$$

so that

$$C\tau^a C^{-1} = -\tau^{aT},$$

and the charge conjugate of a Dirac spinor θ is given by

$$\theta_C = C\theta^{\dagger T}. \tag{A4}$$

Fierz identities follow from the τ-identities

$$\tau^a_{ij}\tau^a_{kl} + \delta_{ij}\delta_{kl} = 2\delta_{il}\delta_{kj}, \tag{A5}$$

$$\tau^a_{ij}\delta_{kl} + \delta_{ij}\tau^a_{kl} = \tau^a_{ij}\delta_{kj} + \delta_{il}\tau^a_{kj}, \tag{A6}$$

$$\epsilon^{abc}\tau^b_{ij}\tau^c_{kl} = i\left(\tau^a_{il}\delta_{kj} - \delta_{il}\tau^a_{kj}\right). \tag{A7}$$

Fierz identities which have been used in this paper are

$$(\theta^\dagger\tau^a\lambda)(\theta^\dagger\theta) = -(\theta^\dagger\tau^a\theta)(\theta^\dagger\lambda), \tag{A8}$$

$$(\xi^\dagger\theta)(\theta^\dagger\theta_{\mathrm{C}}) = -2(\xi^\dagger\theta_{\mathrm{C}})(\theta^\dagger\theta), \tag{A9}$$

$$(\theta^\dagger\tau^a\theta)(\theta^\dagger\tau^a\theta) = -\delta^{ab}(\theta^\dagger\theta)^2 = -\frac{1}{2}\delta^{ab}(\theta^\dagger_{\mathrm{C}}\theta)(\theta^\dagger\theta_{\mathrm{C}}), \tag{A10}$$

$$\theta_\kappa\theta^\dagger_l = -\frac{1}{2}C_{kl}\theta^\dagger\theta_{\mathrm{C}}, \tag{A11}$$

$$\theta^\dagger_k\theta_l = -\frac{1}{2}C_{kl}\theta^\dagger_{\mathrm{C}}\theta, \tag{A12}$$

$$(\Lambda^\dagger\theta)(\xi^\dagger\theta) = \frac{1}{2}(\Lambda^\dagger\xi_{\mathrm{C}})(\theta^\dagger_{\mathrm{C}}\theta), \tag{A13}$$

$$(\theta^\dagger\Lambda)(\theta^\dagger\xi) = \frac{1}{2}(\xi^\dagger_{\mathrm{C}}\Lambda)(\theta^\dagger\theta_{\mathrm{C}}), \tag{A14}$$

$$(\theta^\dagger\Lambda)(\xi^\dagger\theta) = -\frac{1}{2}(\theta^\dagger\tau^a\theta)(\xi^\dagger\tau^a\Lambda) - \frac{1}{2}(\theta^\dagger\theta)(\xi^\dagger\Lambda). \tag{A15}$$

For Grassmann integration we employ

$$\int d^2\theta\,\theta_i\theta_j = -\frac{1}{2}C_{ij} = \int d^2\theta^\dagger\,\theta^\dagger_i\theta^\dagger_j \tag{A16}$$

so that

$$\int d^2\theta\,(\theta^\dagger_{\mathrm{C}}\theta) = 1 = \int d^2\theta^\dagger\,(\theta^\dagger\theta_{\mathrm{C}}). \tag{A17}$$

The bosonic operators P^a and Z are represented by

$$P_a = -i\frac{\partial}{\partial x^a} = -i\nabla_a, \tag{A18}$$

$$Z = -i\frac{\partial}{\partial\zeta} = -i\partial_\zeta. \tag{A19}$$

We also use the notation

$$\nabla_a f(x,\zeta) = f_{,a}, \tag{A20}$$

$$\partial_\zeta f(x,\zeta) = f_{,\zeta}. \tag{A21}$$

For derivatives with respect to Grassman coordinates we use the notation

$$\frac{\partial}{\partial\theta_i} = \partial_i, \tag{A22}$$

$$\frac{\partial}{\partial\theta^\dagger_i} = \partial^\dagger_i. \tag{A23}$$

APPENDIX B

In this appendix we note that, remarkably, there exists a superalgebra associated with 3dE in which $[P^a, P^b] \neq 0$. If we define $\tilde{Q} = Q^T \tau_2$, then the superalgebra

$$\{Q, \tilde{Q}\} = \vec{\tau} \cdot \vec{J}, \qquad \{Q, Q^\dagger\} = \vec{\tau} \cdot \vec{P}, \tag{B1a,b}$$

$$[J^a, Q] = -\frac{1}{2}\tau^a Q, \qquad [P^a, \tilde{Q}] = \frac{1}{2} Q^\dagger \tau^a, \tag{B1c,d}$$

$$[J^a, J^b] = i\epsilon^{abc} J^c, \qquad [P^a, P^b] = -i\epsilon^{abc} J^c, \tag{B1e,f}$$

$$[J^a, P^b] = i\epsilon^{abc} P^c \tag{B1g}$$

satisfies the Jacobi identities. Alternatively, if we set

$$Q_1 = \frac{Q + Q_C}{2} = -(Q_2)_C, \tag{B2a}$$

$$Q_2 = \frac{Q - Q_C}{2} = +(Q_1)_C \tag{B2b}$$

then (B1a,b) becomes

$$\{Q_1, \tilde{Q}_2\} = \{Q_2, \tilde{Q}_1\} = \frac{1}{2}\vec{\tau} \cdot \vec{J}, \tag{B3a}$$

$$\{Q_1, \tilde{Q}_1\} = \{Q_2, \tilde{Q}_2\} = -\frac{1}{2}\vec{\tau} \cdot \vec{P}. \tag{B3b}$$

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